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LETTER TO THE EDITOR

Membrane geometry with auxiliary variables and quadratic constraints

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Abstract
Consider a surface described by a Hamiltonian which depends only on the metric and extrinsic curvature induced on the surface. The metric and the curvature, along with the basis vectors which connect them to the embedding functions defining the surface, are introduced as auxiliary variables by adding appropriate constraints, all of them quadratic. The response of the Hamiltonian to a deformation in each of the variables is determined and the relationship between the multipliers implementing the constraints and the conserved stress tensor of the theory established. For the purpose of illustration, a fluid membrane described by a Hamiltonian quadratic in curvature is considered.

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Geometrical surfaces occur as representations of physical systems across a spectacular range of scales spanning string theory, cosmology, condensed matter and biophysics [1–5]. While the physics they describe may be very different, the models involved share a common feature: the action or Hamiltonian describing the surface is constructed out of simple geometrical invariants of the surface and fields which couple to it. A nice example, close to home, is provided by a fluid membrane consisting of amphiphilic molecules which aggregate spontaneously into bilayers in water; at mesoscopic scales the membrane is described surprisingly well by a Hamiltonian proportional to the integrated square of the mean curvature [7, 8]. A close Lorenzian analogue describes colour flux tubes in QCD [5, 6]. There is now an extensive literature on the field theory of geometrical models of this kind; a good point of entry is provided by the review articles collected in [9, 10].

While the relevant geometrical model itself may be easy to identify, typically it will involve derivatives higher than first and inherit a level of non-linearity from the geometrical invariants of the surface. There is, however, a useful stratagem to lower the effective order or to tame this non-linearity involving the introduction of auxiliary fields. In the description of a surface by a set of embedding functions **X**, the metric induced on the surface is often replaced by an auxiliary intrinsic metric g_{ab} [11, 5]; by amending the Hamiltonian with the appropriate constraints, g_{ab} is freed to be varied independently of **X**.[1]

Whereas the introduction of g_{ab} as an auxiliary variable may be sufficient for the technical purposes originally contemplated—providing a tractable inroad on the evaluation of a functional integral—from a purely geometrical point of view it is natural to question why one should stop with the metric. In this letter, I will explore the possibility of introducing additional auxiliary variables. Is it possible, for example, to treat the extrinsic curvature as an independent geometrical variable? This would be useful in geometric theories involving higher derivatives.

Consider, for simplicity, a hypersurface with a single normal vector. The extrinsic curvature K_{ab} is defined in terms of the behaviour of this vector as it ranges over the surface; together with the metric, it completely characterizes the surface geometry. If K_{ab} could be treated, like g_{ab}, as an auxiliary field, the original theory describing a surface would be replaced by a simple tensor field theory for K_{ab} on a curved space described by g_{ab}. The subtlety, of course, now lies in the implementation of the constraints. The surprise is that it is possible to do this in a way which is not only tractable but also, en route, reveals a structure inherent to any theory of embedded surfaces. Of course, if the constraints themselves were to introduce new non-linearities the value of the exercise would be very limited. This would certainly be the case in their implementation within a functional integral [5]. In this respect, the metric tensor provides a useful set of auxiliary variables because the induced metric depends quadratically on the first derivatives of the **X**. In contrast, as things stand, the constraints involved in K_{ab}'s promotion to auxiliary variable status are not quadratic. Fortunately it is simple to resolve this difficulty: the basis vectors, the normal and the tangents, are themselves introduced as intermediate auxiliary variables and the constraint defining K_{ab} is implemented (not in one but) in a sequence of steps each of which involves a quadratic. In a translationally invariant theory, **X** only appears though the tangent vectors; in such a theory, **X** is now consigned to the constraint defining the tangents and will appear nowhere else.

With the constraints in place it is possible to consider the response of the Hamiltonian to deformations of each of these variables in turn: for **X** the Euler–Lagrange derivative is a divergence; in equilibrium this gives the conservation law associated with translational invariance; the stress tensor gets identified with the multipliers implementing the tangential constraints. The auxiliary variables dispatch the task of constructing this tensor in two clearly defined steps: first, the Euler–Lagrange equations for the basis vectors express it in terms of the remaining multipliers; these multipliers are then fixed by the Euler–Lagrange equations for the metric and the curvature. The procedure is completely independent of the details of the particular model. As described here, its implementation depends in a unvarying way on each of the auxiliary variables.

It is worthwhile to contrast the above picture with the more familiar one which results when the metric alone is treated as an auxiliary variable. The stress tensor coupling to the intrinsic geometry is identified with the Lagrange multipliers implementing the corresponding constraints. However, this tensor will *not* generally be conserved: the induced metric characterizes only the intrinsic geometry of the surface; there remains considerable freedom as to how the surface is embedded in its surroundings. The remaining multipliers capture this missing information permitting the reconstruction of the full conserved stress tensor underlying the geometry. The metric is just one element in the complete description.

[1] Language appropriate to equilibrium statistical mechanics will be used: the Hamiltonian is a functional of **X**.

The geometry of interest is a D-dimensional surface embedded in R^{D+1} described by $\mathbf{x} = \mathbf{X}(\xi^1, \ldots, \xi^D)$. Higher co-dimensions will not be considered though it is straightforward to do so; it is also straightforward to adapt the discussion to consider timelike surfaces in Minkowski space. Indeed, the description may also be extended to surfaces in a curved background. The notation used is $\mathbf{x} = (x^1, \ldots, x^{D+1})$; the parameters $\xi^1, \ldots, \xi^D$ represent local coordinates on the surface. One now shifts the focus of attention from the embedding functions $\mathbf{X}$ to the geometrical tensors induced by them, the metric and the extrinsic curvature (for example, see [12])

$$g_{ab} = \mathbf{e}_a \cdot \mathbf{e}_b \qquad K_{ab} = \mathbf{e}_a \cdot \partial_b \mathbf{n} \tag{1}$$

$a, b = 1, \ldots, D$, where $\mathbf{e}_a$ are tangent and $\mathbf{n}$ is unit normal to the surface:

$$\mathbf{e}_a = \partial_a \mathbf{X} \qquad \mathbf{e}_a \cdot \mathbf{n} = 0 \qquad \mathbf{n}^2 = 1. \tag{2}$$

Together, g_{ab} and K_{ab} encode the geometrically significant derivatives of $\mathbf{X}$; all geometrical invariants, the Hamiltonian included, can be cast as functionals of g_{ab} and K_{ab}.

Consider any reparametrization invariant functional of the variables g_{ab} and K_{ab},

$$H[\mathbf{X}] = \int \mathrm{d}A\, \mathcal{H}(g_{ab}, K_{ab}). \tag{3}$$

The area element is $\mathrm{d}A = \sqrt{\det g_{ab}}\, \mathrm{d}^D\xi$. We are interested in determining the response of H to a deformation of the surface: $\mathbf{X} \to \mathbf{X} + \delta\mathbf{X}$. The approach adopted here will be to distribute the burden on $\mathbf{X}$ among $\mathbf{e}_a$, $\mathbf{n}$, g_{ab} and K_{ab} treating the latter as independent auxiliary variables. To do this consistently the structural relationships connecting the variables must be preserved under the deformation; thus equations (1) defining g_{ab} and K_{ab} in terms of the basis vectors $\mathbf{e}_a$ and $\mathbf{n}$, as well as equations (2) which define these vectors, are introduced as constraints; H is amended accordingly.

Introduce Lagrange multiplier functions to implement the constraints. We thus construct a new functional $H_C[g_{ab}, K_{ab}, \mathbf{n}, \mathbf{e}_a, \mathbf{X}, \mathbf{f}^a, \Lambda^{ab}, \lambda_{ab}, \lambda_\perp^a, \lambda_n]$ as follows:

$$\begin{aligned} H_C = H[g_{ab}, K_{ab}] &+ \int \mathrm{d}A\, \mathbf{f}^a \cdot (\mathbf{e}_a - \partial_a \mathbf{X}) + \int \mathrm{d}A\, \left(\lambda_\perp^a (\mathbf{e}_a \cdot \mathbf{n}) + \lambda_n (\mathbf{n}^2 - 1)\right) \\ &+ \int \mathrm{d}A (\Lambda^{ab}(K_{ab} - \mathbf{e}_a \cdot \partial_b \mathbf{n}) + \lambda^{ab}(g_{ab} - \mathbf{e}_a \cdot \mathbf{e}_b)). \end{aligned} \tag{4}$$

Note that the original Hamiltonian H is now treated as a function of the independent variables, g_{ab} and K_{ab} but not of $\mathbf{e}_a$, $\mathbf{n}$ or $\mathbf{X}$. The multiplier $\mathbf{f}^a$ anchors $\mathbf{e}_a$ to the embedding $\mathbf{X}$; it is simultaneously a spatial vector and a surface vector. Its geometrical character is dictated by the constraint it imposes. Likewise, the multipliers Λ^{ab} and λ^{ab} are symmetric surface tensors; $\lambda_\perp^a$ is a surface vector and λ_n is a scalar. We are now free to treat g_{ab}, K_{ab}, $\mathbf{n}$, $\mathbf{e}_a$ and $\mathbf{X}$ as independent variables which can be deformed independently. It is not necessary to track explicitly the deformation induced on g_{ab} and K_{ab} by a deformation in $\mathbf{X}$.

The only place where $\mathbf{X}$ appears explicitly in H_C is within the constraint which defines $\mathbf{e}_a$. The corresponding Euler–Lagrange derivative is a divergence

$$\delta H_C / \delta \mathbf{X} = \nabla_a \mathbf{f}^a. \tag{5}$$

In this expression ∇_a is the symmetric covariant derivative compatible with g_{ab} and operates on surface indices. In equilibrium, $\mathbf{f}^a$ is covariantly conserved on the surface. The physical interpretation of $\mathbf{f}^a$ as a stress tensor will be commented on below.

The Euler–Lagrange equations for $\mathbf{e}_a$ express the conserved 'vector' $\mathbf{f}^a$ as a linear combination of the basis vectors:

$$\mathbf{f}^a = \left(\Lambda^{ac} K_c{}^b + 2\lambda^{ab}\right)\mathbf{e}_b - \lambda_\perp^a \mathbf{n}. \tag{6}$$

The Weingarten equations $\partial_a \mathbf{n} = K_a{}^b \mathbf{e}_b$ have been used to obtain equation (6). They themselves follow from the constraints on K_{ab} and the normalization of $\mathbf{n}$. Remarkably, $\mathbf{f}^a$ is determined in a model independent way in terms of the Lagrange multipliers imposing the geometrical constraints. The values assumed by the multipliers will, of course, depend on the specific Hamiltonian H.

The multiplier $\lambda_\perp^a$ enforcing orthogonality appearing in equation (6) is fixed by the Euler–Lagrange equation for $\mathbf{n}$. Using the Gauss equations $\nabla_a \mathbf{e}_b = -K_{ab}\mathbf{n}$ (which themselves follow from the Weingarten equations and the orthogonality constraint), one has

$$\left(\nabla_b \Lambda^{ab} + \lambda_\perp^a\right)\mathbf{e}_a + (2\lambda_n - \Lambda^{ab} K_{ab})\mathbf{n} = 0 \tag{7}$$

and thus

$$\lambda_\perp^a = -\nabla_b \Lambda^{ab} \tag{8}$$

$$2\lambda_n = \Lambda^{ab} K_{ab}. \tag{9}$$

$\lambda_\perp^a$ is identified as (minus) the divergence of Λ^{ab}; the normal component of $\mathbf{f}^a$ will generally involve one derivative more than its tangential components. Note that λ_n does not appear in the stress tensor. This is not surprising: the role of λ_n is to enforce the normalization of $\mathbf{n}$, which is important for reasons of mathematical consistency but not physically.

The missing ingredients are the multipliers Λ^{ab} and λ^{ab} appearing in the tangential part of $\mathbf{f}^a$. They are determined by the Euler–Lagrange equations for K_{ab} and g_{ab}:

$$\Lambda^{ab} = -\mathcal{H}^{ab} \tag{10}$$

$$\lambda^{ab} = T^{ab}/2 \tag{11}$$

where $\mathcal{H}^{ab} = \partial\mathcal{H}/\partial K_{ab}$ and $T^{ab} = -2(\sqrt{g})^{-1}\partial(\sqrt{g}\mathcal{H})/\partial g_{ab}$ is the intrinsic stress tensor associated with the metric g_{ab}. The conserved stress $\mathbf{f}^a$ is

$$\mathbf{f}^a = \left(T^{ab} - \mathcal{H}^{ac} K_c{}^b\right)\mathbf{e}_b - \nabla_b \mathcal{H}^{ab}\mathbf{n}. \tag{12}$$

Note that T^{ab} is only one part of the total stress tensor, and it is entirely tangential; it is not generally conserved.

There is no difficulty treating a Hamiltonian of the more general form $\mathcal{H}(g_{ab}, K_{ab}, \nabla_a K_{bc}, \ldots)$ within this framework; the derivatives appearing in T^{ab} and $\mathcal{H}^{ab}$ are simply replaced by functional derivatives. It is also unnecessary to consider an explicit intrinsic curvature dependence in $\mathcal{H}$. This is because the Gauss–Codazzi equations [12]

$$R_{abcd} = K_{ac}K_{bd} - K_{ad}K_{bc} \tag{13}$$

completely fix the Riemann tensor in terms of the extrinsic curvature.

Now let us look at a few examples. For a soap film, or a Dirac–Nambu–Goto membrane, H is proportional to the surface area with a constant surface tension μ: $\mathcal{H}^{ab} = 0$, and $T^{ab} = -\mu g^{ab}$; the stress is determined completely by the metric; the only relevant constraints are intrinsic. A less simple example is provided by the Helfrich Hamiltonian without adornment describing a fluid membrane with $\mathcal{H} = \alpha K^2 + \mu$ in equation (3), where $K = g^{ab}K_{ab}$. The first term, a conformal invariant when $D = 2$, was introduced by Willmore[13]. One has $\mathcal{H}^{ab} = 2\alpha g^{ab} K$, and $T^{ab} = \alpha K(4K^{ab} - K g^{ab}) - \mu g^{ab}$. Thus

$$\mathbf{f}^a = [\alpha K(2K^{ab} - K g^{ab}) - \mu g^{ab}]\mathbf{e}_b - 2\alpha\nabla^a K\mathbf{n}. \tag{14}$$

In general, if $\mathcal{H}$ does not involve derivatives of K_{ab}, as is the case in the description of a fluid membrane, neither will Λ^{ab} or λ^{ab}. Thus the tangential component of $\mathbf{f}^a$ will not involve derivatives of curvatures.

Equation (5) casts the Euler–Lagrange equations for $\mathbf{X}$ as a conservation law, $\nabla_a \mathbf{f}^a = 0$. Following [16], write

$$\mathbf{f}^a = f^{ab}\mathbf{e}_b + f^a\mathbf{n}. \tag{15}$$

The projections of the conservation law normal and tangent to the surface give, respectively:

$$\nabla_a f^a - K^{ab} f_{ab} = 0 \tag{16}$$

$$\nabla_a f^{ab} + K^{ab} f_a = 0. \tag{17}$$

Equation (16) is the 'shape' equation. For the example considered above, it reads [15]

$$-2\alpha\nabla^2 K - \alpha K K^{ab}(2K_{ab} - K g_{ab}) + \mu K = 0. \tag{18}$$

Because H is invariant under reparametrizations, the only physical deformations are those normal to the surface. There is a single 'shape' equation [16]. Equations (17) are consistency conditions on the components of the stress tensor. For a Hamiltonian invariant under reparametrizations, they reduce to simple geometrical identities.

This framework also provides a physical interpretation of the conserved multiplier $\mathbf{f}^a$. Look at the divergence that was legitimately discarded in the derivation of the Euler–Lagrange equations: modulo these equations, the deformed Hamiltonian is

$$\delta H_C = -\int \mathrm{d}A\, \nabla_b(\Lambda^{ab}\mathbf{e}_a \cdot \delta\mathbf{n} + \mathbf{f}^b \cdot \delta\mathbf{X}). \tag{19}$$

A spatial translation $\delta\mathbf{x} = \mathbf{a}$, where $\mathbf{a}$ is some constant vector, induces the internal symmetry $\delta\mathbf{X} = \mathbf{a}$; all of the other variables are unchanged. In particular, $\delta\mathbf{n} = 0$ in equation (19). Thus

$$\delta H_C = -\mathbf{a} \cdot \int \mathrm{d}A\, \nabla_a \mathbf{f}^a. \tag{20}$$

On a domain Σ with boundary the left-hand side may be cast as an integral over this boundary; the vector $\eta_a \mathbf{f}^a\, \mathrm{d}S$, where η_a is the outward normal to the boundary $\partial\Sigma$, is thus identified as the force on the boundary element $\mathrm{d}S$ due to the action of the stresses $\mathbf{f}^a$ set up within the domain.

The construction of the stress tensor for a fluid membrane was considered some time ago by Evans in a bio-mechanical context [14]. In [16], the problem was reconsidered from a geometrical perspective, and the stress tensor identified as the conserved Noether current associated with translational invariance. This was done by tracking the response of the metric and extrinsic curvature to the deformation in the embedding functions. The approach via auxiliary variables, adopted here, has the virtue of sidestepping the need to know how g_{ab} or K_{ab} themselves respond to a deformation in $\mathbf{X}$ and the attendant problem of doing so in a way which respects the invariance under change of parametrization.

A few technical comments on the choice of constraints:

(1) All of the constraints are bi-linear in the vectors $\mathbf{e}_a$ and $\mathbf{n}$ with one exception—the linear constraint, $\mathbf{e}_a = \partial_a \mathbf{X}$. It would be consistent to implement the linear Gauss–Weingarten equations as vector constraints in place of the bilinear definition of K_{ab}. There is, however, a sound reason not to: with the bi-linear choice of constraint used here, $\mathbf{f}^a$ gets identified directly as a linear combination of $\mathbf{e}_a$ and $\mathbf{n}$.

(2) It is consistent to use a reduced set of auxiliary variables; for example, the tangent vectors $\mathbf{e}_a$ or the curvatures K_{ab} could be dropped. In the former case, instead of implementing $\mathbf{e}_a = \partial_a \mathbf{X}$ as a constraint, substitute in favour of $\mathbf{X}$ everywhere $\mathbf{e}_a$ appears. If one also drops K_{ab} as an independent variable, then $K^2 = (\nabla^2\mathbf{X})^2$ [11]; for the Helfrich Hamiltonian, $\mathbf{n}$ does not appear so it is also consistent to drop the constraints involving $\mathbf{n}$. The only

remaining auxiliary variable is the metric: the original auxiliary variable inspiring this generalization. The disadvantage of this truncation is that the constraint $\mathbf{e}_a = \partial_a \mathbf{X}$ comes with the marker identifying the stress tensor and, when it is dropped, with it goes the conservation law encoded in (5) as well as the identification of the conserved stress tensor appearing within it.

(3) If one attempts to treat g_{ab}, K_{ab}, $\mathbf{n}$, $\mathbf{e}_a$ and $\mathbf{X}$ as independent variables with an insufficient set of constraints, an inconsistent set of equations is usually obtained. For example, had the normalization constraint been dropped, instead of identifying λ_n, equation (9) would have given $\Lambda^{ab} K_{ab} = 0$ which is nonsense unless, of course, Λ^{ab} itself turns out to be zero—as it does for the soap film.

(4) There is no need to implement the Gauss–Codazzi, or the Codazzi–Mainardi integrability conditions explicitly as constraints. Recall that the former are given by equations (13); the latter are $\nabla_a K_{bc} - \nabla_b K_{ac} = 0$. When the constraints appearing in equation (4) are satisfied, the integrability conditions are automatically accounted for. One might choose to focus, however, on a specific parametrization of g_{ab} or K_{ab} (asymptotic coordinates, for example) which is not anchored to a specific embedding. In such a case, consistency would require the implementation of the integrability conditions as additional constraints.

To conclude, a geometrical framework involving auxiliary variables has been introduced to examine a theory of surfaces described by a reparametrization invariant Hamiltonian, exemplified by the Helfrich Hamiltonian describing fluid membranes. For this Hamiltonian, the only variable which appears in a non-quadratic way in H_C is the metric. This would suggest that the approach has the potential to provide novel approximations to geometrical functional integrals. Because of the central role played by the stress tensor, it should also prove useful in the study of membrane-mediated interactions [17]; this would certainly appear to be the case if non-perturbative effects are important and it becomes necessary to look beyond the quadratic truncation of the Hamiltonian in terms of the height function. By implementing geometrical constraints using Lagrange multipliers, it is possible to establish useful connections between models for embedded surfaces and other, more fully studied or more tractable, models. For example, it is possible to consider the Helfrich model as a constrained $O(D+1)$ non-linear sigma model on the surface [5]. The Hamiltonian density is $(\nabla_a \mathbf{n})^2$ subject to the constraint $\mathbf{e}_a \cdot \mathbf{n} = 0$ on the unit vector. Applications, as well as generalizations, will be considered in forthcoming publications.

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